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Common Fixed Point Theorems for G-Non Decreasing Maps Satisfying Generalized Condition (B) In Ordered G-Metric Spaces

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Abstract: **In this paper, we obtain common fixed point theorems for g – non decreasing maps satisfying generalized condition (B) in partially ordered G- metric spaces. These results generalize the results of Agarwal and Karpinar [25]. We also, give examples that our results are not equivalent to the results in metric spaces if we transform a metric space into a G- metric space.**

Keywords: **Partially ordered sets, generalized condition (B), g – non decreasing maps, common Fixed points, G-metric spaces.**

1. INTRODUCTION

In 2004, Mustafa and Sims [20] introduced the concept of G- metric space as a generalization of metric space and obtained Banach contraction mapping theorem in the context of G- metric spaces . Since then several authors obtained several fixed point results in these spaces . Some of these works are noted in [2-6,14,16,17,21,22,23,25,27,30].

On the other hand, Ran and Reurings [25] proved the existence and uniqueness of fixed point of a contraction mapping in partially ordered metric spaces. Following this initial work, many authors worked in this direction and investigated fixed points of various mappings. For more literature in this direction, we refer[7-13,18,19,24,28,29].

Before starting our main results we recall some basic definitions and existing theorems.

Definition 1.1: [20]. Let X be a nonempty set, $G: X \times X \times X \to R^+$ be a mapping satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$.

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ for all $x, y, z \in X$ ((symmetry in all variables).

(G5) $G(x, y, z) \leq G(x, a, a) + G (a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality)

Then the mapping G is called a *generalized metric*, or, more specially, a *G- metric on X*, and the pair (X, G) is called a *G- metric space.*

Every G- metric on X defines a metric d_G on X and it is given by $d_G(x, y) = G(x, x, y) + G(y, x, x)$ for all x, y in X.

Example 1.2:[20]. Let (X, d) be a metric space. Then the function $G: X \times X \times X \to R^+$ defined by

$G(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$

Or

$G(x, y, z) = d(x, y) + d(y, z) + d (z, x)$,

for all $x, y, z \in X$ is a G- metric space on X.

Definition 1.3:[20]. Let (X, G) be a G metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is *G-convergent* to $x \in X$ if $\lim_{n\to\infty} G(x, x_n, x_m) = 0$. That is, for any $\in > 0$, there exists $N \in \mathbb{Z}^+$ such that $G(x, x_n, x_m) < \in$ for all $n, m \geq N$. We call x the limit of the sequence and we write $x_n \xrightarrow{G} x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.4: [20]. In a G- metric space (X, G), the following are equivalent:

(i) $\{x_n\}$ is G- convergent to x

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$.

(iv) $G(x_n, x_m, x) \to 0$ as $n \to \infty$.

Definition 1.5: [20]. Let (X, G) be a G- metric space. A sequence $\{x_n\}$ is called a *G- Cauchy sequence*

if , for any $\in > 0$, there is $N \in \mathbb{Z}^+$ such that $G(x_n, x_m, x_l) < \in$ for all $m, n, l \geq N$ that is

$G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.6: [20]**.** In a G- metric space (X, G), the following are equivalent:

(1) The sequence $\{x_n\}$ is G- Cauchy

(2) for any $\in > 0$, there exists $N \in \mathbb{Z}^+$ such that $G(x_m, x_n, x_n) < \in$ for all $n, m \geq N$.

Definition 1.7: **[**20]**.** A G- metric space (X, G) is called *G- complete* if for every G- Cauchy sequence is G- convergent in (X, G).

Lemma 1.8: [20]. Every G- convergent sequence in (X, G) is G -Cauchy sequence

The converse of Lemma 1.8 need not be true, *i.e.*, every G- Cauchy sequence need not be G-convergent in (X, G).

Remark 1.9:[20]. Let (X, G) be a metric space .Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition 1.10: [20]. A G- metric space (X, G) is said to be *symmetric* if G(x, y, y) = G(y, x, x) for all $x, y \in X$.

Next we give an example of non-symmetric metric spaces.

Example 1.11: [20]. Let X={a,b} , let G(a,a,a)=G(b, b,b)=0, G(a,a,b)=1, G(a,b,b)=2 and extend G to all of $X x X x X$ by symmetry in the variables. Then G is a G-metric. It is non-symmetric since G(a,b,b)≠ G(a,a,b).

Definition 1.12: [12]. Let (X, ≼) be a partially ordered set. Let f, g: X → X be two self maps on X. We say that f is *g-nondecreasing*(resp., *g-nonincreasing*) if

$$gx \preccurlyeq gy \text{ implies } fx \preccurlyeq fy. \quad (\text{rep., } gx \preccurlyeq gy \text{ implies } fx \succcurlyeq fy)$$

The results of Ciric[12] were extended by Agrawal and Karpinar[25](Corollary 3.4) in G- metric space setting, which are stated below.

Theorem1.13:[25]. *Let (X, ≼) be a partially ordered set and suppose that there is a G metric on X such that (X , G) is a complete G- metric space . Let f, g: X → X be two self maps on X satisfying the following the conditions:*

(*i*) $f(X) \subseteq g(X)$;

(*ii*) *f is g – non decreasing mapping(with respect to ≼);*

(*iii*) (*a*) *If for any non decreasing sequence* $\{x_n\}$ *in X with* $x_n \to z$ *then* $gx_n \preccurlyeq gz$, *for all n ;*

(*b*) *If for any non increasing sequence* $\{y_n\}$ *in X with* $y_n \to z$ *then* $gy_n \succcurlyeq gz$, *for all n ;*

(*iv*) *There exists a constant* $\alpha \in (0, 1)$ *such that for all x ,y z ∈ X with* $gx \succcurlyeq gy \succcurlyeq gz$

$$G(fx, fy, fz) \leq \alpha\, G(gx, gy, gz); \tag{1.13.1}$$

(*v*) *There exists an* $x_0 \in X$ *with* $gx_0 \preccurlyeq fx_0$.

(*vi*) *g is G-continuous and commute with f;*

Then f and g have a coincidence point, i.e., there exists $z \in X$ such that fz=gz.

Recently, Golubovic et. al. [15] proved the following fixed point theorem in ordered metric spaces.

Theorem 1.14: [15]. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a d metric on X such that (X, d) is a complete metric space . Let $f, g: X \to X$ be two self maps on X satisfying the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) *$g(X)$ is complete ;*

(iii) *f is g – non decreasing mapping;*

(iv) *there exists a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$ with $gx \succcurlyeq gy$*

$$d(fx, fy) \leq \alpha \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\} ;$$

(v) *there exists an $x_0 \in X$ with $gx_0 \preccurlyeq fx_0$;*

(vi) $\{g(x_n)\} \subseteq X$ *is a non decreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then*

$g x_n \preccurlyeq g z$, $g z \preccurlyeq g(g z)$ *for all n holds.*

Then f and g have a coincidence point, i.e., there exists $z \in X$ such that fz=gz. Further, if f and g are weakly compatible, then f and g have a common fixed point.

Recently, Babu, *et.al.*, [1] introduced the following class of mappings satisfying an almost generalized contractive condition.

Definition 1.15: [1]. Let (X, d) be a metric space, and let $f, g : X \to X$. A mapping g is called an '*generalized condition* (*B*)' if there exist $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(fx,fy) \leq \delta M(x, y) + L \min\{ d(fx, gx), d(fy,gy), d(fx,gy), d(fy, gx)\}$$

for all $x, y \in X$, where $M(x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx,gy)+ d(fy,gx)}{2}\}$.

Motivated, by the above results in this paper, we define 'generalized condition (B)' in G-metric spaces ((condition (2.1.1)). We obtain unique common fixed point theorems for g – non decreasing mappings satisfying 'generalized condition (B)' in partially ordered G- metric spaces. These results generalise the results of Agarwal and Karpinar [25] . We also, give examples that our results are not equivalent to the results in metric spaces if we transform a metric space into a G- metric space.

2. MAIN RESULTS

Theorem 2.1: *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a G metric on X such that (X, G) is a complete G- metric space . Let f and g be two self maps on X satisfying the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) *$g(X)$ is closed*;

(iii) *f is g – non decreasing mapping;*

(iv) *there exists a constant $\alpha \in (0, 1)$ such that for all $x, y, z \in X$ with $gx \succcurlyeq gy \succcurlyeq gz$,*

$$G(fx, fy, fz) \leq \alpha \max\{G(fx, gy, gz), G(gx, fy, gz), G(gx, gy, gz), G(gx, gy, fz), G(gx, fx, fx), G(fx, fy, gx)\} + L \min\{G(fx, fx, gz), G(gx, fx, gz), G(gx, gy, fx)\} \quad (2.1.1)$$

(v) there exists an $x_0 \in X$ with $g x_0 \preccurlyeq f x_0$;

(vi) $\{g(x_n)\} \subseteq X$ is a non decreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then

$g x_n \preccurlyeq gz$, $gz \preccurlyeq g(gz)$ for all n holds.

Then f and g have a coincidence point, i.e., there exists $z \in X$ such that fz=gz. Further, if f and g are weakly compatible, then f and g have a common fixed point.

Proof: Let $x_0 \in X$. Since $g x_0 \preccurlyeq f x_0$, $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $g x_1 = f x_0$, this implies

$$g x_0 \preceq g x_1 . \tag{2.1.2}$$

By using the property of f is g- non decreasing, we have

$$f x_0 \preccurlyeq f x_1 . \tag{2.1.3}$$

Again, we can choose x_2 in X such that $g x_2 = f x_1$. On continuing this process, we can choose $\{y_n\}$ in X such that

$$g x_{n+1} = f x_n = y_n , \text{ for all } n > 0 \tag{2.1.4}$$

Since $f x_0 \preccurlyeq f x_1$, this implies $g x_1 \preccurlyeq g x_2$ this implies $f x_1 \preccurlyeq f x_2$.

On continuing this process, we have $g x_0 \preccurlyeq g x_1 \preccurlyeq g x_2 \preccurlyeq g x_3 \preccurlyeq \ldots g x_n \preccurlyeq g x_{n+1} \preccurlyeq \ldots$. (2.1.5)

We now claim that $\{y_n\}$ is a G- Cauchy sequence in X.

First we suppose that $g x_{n0} = g x_{n0+1}$, for some $n_0 \in N$. This implies $g x_{n0} = f x_{n0}$.

Then f and g have a coincidence point. This completes the proof of the theorem.

Hence suppose that $g x_n \neq g x_{n+1}$ for all $n \in N$.

To prove $\{y_n\}$ is a G-Cauchy first we prove that

$$G(f x_n, f x_{n+1}, f x_{n+1}) \leq G(f x_{n+1}, f x_n, f x_n) \text{ for all } n \in N . \tag{2.1.6}$$

Using (2.1.2), (2.1.4) and (2.1.5), it follows that $g x_n \preccurlyeq g x_{n+1} \preccurlyeq g x_{n+1}$.

Now using (2.1.1), we have

$$\begin{aligned} G(f x_n, f x_{n+1}, f x_{n+1}) \leq \alpha \max\{&G(f x_n, f x_n, f x_n), G(f x_{n-1}, f x_{n+1}, f x_n), G(f x_{n-1}, f x_n, f x_n), \\ &G(f x_{n-1}, f x_n, f x_{n+1}), G(f x_{n-1}, f x_n, f x_n), G(f x_n, f x_{n+1}, f x_{n+1})\} \\ &+L \min\{G(f x_n, f x_n, f x_n), G(g x_{n+1}, f x_n, g x_n), G(g x_n, g x_n, f x_n)\} \end{aligned}$$

by virtue of $\alpha \in (0, 1)$,we have

$$G(f x_n, f x_{n+1}, f x_{n+1}) \leq \alpha \max\{G(f x_n, f x_{n-1}, f x_{n+1}), G(f x_{n-1}, f x_n, f x_n)\} . \tag{2.1.7}$$

If $\max\{G(f x_n, f x_{n-1}, f x_{n+1}), G(f x_{n-1}, f x_n, f x_n)\} = G(f x_n, f x_n, f x_{n-1})$ then from (2.1.7), it follows that

$$G(f x_n, f x_{n+1}, f x_{n+1}) \leq \alpha G(f x_n, f x_n, f x_{n-1}).$$

Hence $$G(f x_n, f x_{n+1}, f x_{n+1}) \leq G(f x_n, f x_n, f x_{n-1}), \tag{2.1.8}$$
thus (2.1.7) holds.

If $\max\{G(f x_n, f x_{n-1}, f x_{n+1}), G(f x_{n-1}, f x_n, f x_n)\} = G(f x_{n-1}, f x_n, f x_{n+1})$, we have

$$\begin{aligned} G(f x_n, f x_{n+1}, f x_{n+1}) &\leq \alpha G(f x_{n-1}, f x_n, f x_{n+1}) \\ &\leq \alpha [G(f x_{n-1}, f x_n, f x_n) + G(f x_n, f x_n, f x_{n+1})] \\ &\leq \alpha [G(f x_{n-1}, f x_n, f x_n) + 2 G(f x_n, f x_{n+1}, f x_{n+1})]. \end{aligned} \tag{2.1.9}$$

This implies $(1-2\alpha) G(f x_n, f x_{n+1}, f x_{n+1}) \leq \alpha G(f x_{n-1}, f x_n, f x_n)$.

Hence, $$G(f x_n, f x_{n+1}, f x_{n+1}) \leq \frac{\alpha}{1-2\alpha} G(f x_{n-1}, f x_n, f x_n). \tag{2.1.10}$$

Thus (2.1.7) holds when $\alpha \in (0, 1/3)$.

Hence $G(f x_n, f x_{n+1}, f x_{n+1}) \leq G(f x_n, f x_n, f x_{n-1})$ for all n.

Thus $\{G(f x_n, f x_{n+1}, f x_{n+1})\}$ is a non decreasing sequence and converges to a limit l (say).

i.e., $\lim_{n\to\infty} G\,(f\,x_n,\ f\,x_{n+1}, f\,x_{n+1}) = l$. (2.1.11)

We now claim that $l = 0$. Suppose that $l > 0$.

Now from (G5) and (2.1.6), we have

$$G(f\,x_{n-1}, f\,x_n, f\,x_{n+1}) \le [\,G(f\,x_{n-1}, f\,x_n, f\,x_n) + G(f\,x_n, f\,x_n, f\,x_{n+1})\,]$$
$$\le [\,G(f\,x_{n-1}, f\,x_n, f\,x_n) + 2\,G(f\,x_n, f\,x_{n+1}, f\,x_{n+1})\,]$$
$$\le [\,G(f\,x_{n-1}, f\,x_n, f\,x_n) + 2\,G(f\,x_{n-1}, f\,x_n, f\,x_n)\,],$$

so that $\frac{1}{3}\ G(f\,x_{n-1}, f\,x_n, f\,x_{n+1}) \le G(f\,x_n, f\,x_n, f\,x_{n-1})$.

Taking limit supremum as $n \to \infty$, and using (2.1.11), we have

$$\frac{1}{3}\ \lim_{k\to\infty} \sup G(f\ x_n, fx_{n-1}, fx_{n+1}) \le l. \tag{2.1.12}$$

Suppose that $\frac{1}{3}\ \lim_{k\to\infty} \sup G(f\ x_n, fx_{n-1}, fx_{n+1}) = \rho$. (2.1.13)

Then from (2.1.12), we have

$$0 \le \rho \le l. \tag{2.1.14}$$

Taking limit supremum as $n \to \infty$, in (2.1.7) , we have

$$l \le \alpha \max\{l, 3\rho\} \tag{2.1.15}$$

If $\max\{\,l, 3\rho\} = 3\rho$, then we have $l \le \alpha\, 3\rho$.

Case (i):If $\alpha \in (0, 1/3)$, then we have

$$l \le 3\,\alpha\,\rho < \rho,$$

a contradiction to (2.1.4).

Case (ii): If $\alpha = 1/3$, then we have $l \le \rho$

a contradiction to (2.1.4).

Case (iii): If $\alpha \in (1/3, 1)$, we have

$$l \le 3\,\alpha\,\rho \le 3\,\alpha\, l,$$

so that $(1 - 3\,\alpha)\ l \le 0$, then $l \le 0$,

a contradiction to $l > 0$. Hence $\max\{\,l,\ 3\,\rho\} = l$.

Therefore (2.1.15), we have $l \le \alpha l <$, a contradiction.

By considering all the above cases, we have $l = 0$.

Hence $\lim_{k\to\infty} G(f\ x_n, fx_{n+1}, fx_{n+1}) = 0$. (2.1.16)

Now, we prove that $\{f\,x_n\}$ is a G- Cauchy sequence.

Suppose that $\{f\,x_n\}$ is not a G-Cauchy sequence. Then there exist $\in > 0$, and two sequences of integers $\{m\,(k)\}$ and $\{n\,(k)\}$ such that for each k natural number k, $m(k) \ge n(k) \ge k$,

$$G(f\,x_{m(k)}, f\,x_{m(k)}, f\,x_{n(k)}) \ge \in. \tag{2.1.17}$$

Corresponding to n (k), the number m (k) is chosen to be the smallest number for which

(2.1.17) holds.

Therefore $G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)}) < \epsilon$. (2.1.18)

From (G 5) and (2.1.17), we have

$$\epsilon \le G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)})$$
$$\le G(f x_{n(k)}, f x_{m(k)-1}, f x_{m(k)-1}) + G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)})$$
$$< \epsilon + G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)}).$$

Letting $k\to\infty$, using (2.1.16), we have

$$\epsilon \le \lim_{k\to\infty} G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)}) \le \epsilon.$$

Therefore, $\lim_{k\to\infty} G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)}) = \epsilon$. (2.1.19)

Again, from (G5), (2.1.17) and (2.1.18), we have

$$\epsilon \le G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)})$$
$$\le G(f x_{n(k)}, f x_{m(k)-1}, f x_{m(k)-1}) + G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)})$$
$$< G(f x_{m(k)}, f x_{m(k)}, f x_{m(k)-1}) + \epsilon.$$

Letting $k\to\infty$, using (2.1.16), we have

$$\epsilon \le \lim_{k\to\infty} G(f x_{n(k)}, f x_{m(k)-1}, f x_{m(k)-1}) \le \epsilon.$$

Therefore, $\lim_{k\to\infty} G(f x_{n(k)}, f x_{m(k)-1}, f x_{m(k)-1}) = \epsilon$. (2.1.20)

Using rectangular inequality (G5), we have

$G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)-1}) \le G(f x_{n(k)-1}, f x_{n(k)}, f x_{n(k)}) + G(f x_{n(k)}, f x_{m(k)-1}, f x_{m(k)-1})$.

Letting $k\to\infty$, using (2.1.16), we have

$$\lim_{k\to\infty} G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)-1}) \le \epsilon. \quad (2.1.21)$$

Again from (G 5), we have

$$G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)}) \le G(f x_{n(k)}, f x_{n(k)-1}, f x_{n(k)-1}) + G(f x_{n(k)-1}, f x_{m(k)-1}, f x_{m(k)-1})$$
$$\le 2G(f x_{n(k)}, f x_{n(k)}, f x_{n(k)-1}) + G(f x_{n(k)-1}, f x_{m(k)-1}, f x_{m(k)-1})$$
$$\le \epsilon + 2G(f x_{n(k)}, f x_{n(k)}, f x_{n(k)-1}).$$

Letting $k\to\infty$, using (2.1.16) and (2.1.21), we get

$$\epsilon \le \lim_{k\to\infty} G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)-1}) \le \epsilon.$$

Hence, $\lim_{k\to\infty} G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)-1}) = \epsilon$. (2.1.22)

On using (G5), we have

$$G(f x_{m(k)}, f x_{m(k)-1}, f x_{n(k)-1}) \le G(f x_{n(k)-1}, f x_{m(k)-1}, f x_{m(k)-1}) + G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{m(k)})$$
$$\le G(f x_{n(k)-1}, f x_{m(k)-1}, f x_{m(k)-1}) + 2G(f x_{m(k)}, f x_{m(k)-1}, f x_{m(k)}).$$

Letting $k\to\infty$, using (2.1.16) and (2.1.20), we have

$$\lim_{k\to\infty} G(f x_{m(k)}, f x_{m(k)-1}, f x_{n(k)-1}) \le \epsilon. \quad (2.1.23)$$

Also,

$G(f x_{n(k)-1}, f x_{m(k)-1}, f x_{m(k)-1}) \le G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)}) + G(f x_{m(k)}, f x_{m(k)-1}, f x_{n(k)-1})$.

Letting $k\to\infty$, using (2.1.16), (2.1.22) and (2.1.23) we have

$$\epsilon \le \lim_{k\to\infty} G(f x_{n(k)-1}, f x_{m(k)}, f x_{m(k)-1}) \le \epsilon$$

so that $\lim_{k\to\infty} G(f x_{n(k)-1}, f x_{m(k)}, f x_{m(k)-1}) = \in$. (2.1.24)

Now on using rectangular inequality G(5), we have

$G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)-1}) \leq G(f x_{n(k)-1}, f x_{m(k)-1}, f x_{m(k)-1}) + G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)})$,

On taking limits as $k \to \infty$, using (2.1.16), and (2.1.22) we get

$$\lim_{k\to\infty} G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)-1}) \leq \in. \quad (2.1.25)$$

From G(5), we have

$G(f x_{n(k)-1}, f x_{m(k)}, f x_{m(k)-1}) \leq G(f x_{n(k)-1}, f x_{m(k)}, f x_{m(k)}) + G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)})$.

Letting $k\to\infty$, using (2.1.16), (2.1.22) and (2.1.25), we get

$$\lim_{k\to\infty} G(f x_{n(k)-1}, f x_{m(k)}, f x_{m(k)}) = \in. \quad (2.1.26)$$

Now on using the inequality (2.1.1) with $x = x_{m(k)}$, $y = x_{m(k)}$, $z = x_{n(k)}$, we have

$$\begin{aligned} G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)}) \leq \alpha \max\{ & G(f x_{m(k)}, f x_{m(k)-1}, f x_{n(k)-1}), G(f x_{m(k)-1}, f x_{m(k)}, f x_{n(k)-1}), \\ & G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)}), G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)-1}), \\ & G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)}) + L \min\{G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)-1}), \\ & G(f x_{m(k)-1}, f x_{m(k)}, f x_{n(k)-1}), G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{m(k)})\} \\ \leq \alpha \max\{ & G(f x_{m(k)}, f x_{m(k)-1}, f x_{n(k)-1}), G(f x_{m(k)-1}, f x_{m(k)}, f x_{n(k)-1}), \\ & G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)}), G(f x_{m(k)-1}, f x_{m(k)-1}, f x_{n(k)-1}), \\ & G(f x_{m(k)-1}, f x_{m(k)}, f x_{m(k)})\} + L \min\{G(f x_{m(k)}, f x_{m(k)}, f x_{n(k)-1}), \\ & G(f x_{m(k)-1}, f x_{m(k)}, f x_{n(k)-1}), 2G(f x_{m(k)}, f x_{m(k)-1}, f x_{m(k)})\}. \end{aligned}$$

Letting $k\to\infty$, using (2.1.16), (2.1.19), (2.1.20), (2.1.22), (2.1.24), (2.1.26), we get

$$\in \leq \alpha \max\{\in, \in, \in, \in, \in, 0\} + L\min\{\in, \in, 0\} = \alpha \in,$$

a contradiction since $\alpha \in (0, 1)$.

Therefore $\{f x_n\}$ is a G-Cauchy sequence in X. Hence there exists ω such that $\lim_{n\to\infty} f x_n = \omega$.

On using (2.1.4), we get $\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_{n+1} = \omega$. (2.1.27)

By the assumption of g (X) is closed, there exists $u \in X$ such that $\omega = g u$. Hence $\lim_{n\to\infty} g x_{n+1} = \lim_{n\to\infty} f x_n = gu$.

We now show that g u (=ω) is a common fixed point of f and g.

Since $\{g x_n\}$ is non decreasing and $g x_n \to g u$ as $n\to\infty$, by assumption (vi), it follows that

$g x_n \preccurlyeq g u$ for all n.

On using the inequality (2.1.1) with $x = x_n$, $y = x_n$ and $z = u$, we have

$$\begin{aligned} G(f x_n, f x_n, fu) \leq \alpha \max\{ & G(f x_n, g x_n, g u), G(g x_n, f x_n, g u), G(g x_n, g x_n, g u), \\ & G(g x_n, g x_n, fu), G(g x_n, f x_n, f x_n), G(f x_n, f x_n, g x_n)\} \\ & + L \min\{G(f x_n, g x_n, g u), G(g x_n, f x_n, g u), (g x_n, g x_n, f x_n)\}. \end{aligned}$$

Letting $n\to\infty$, using (2.1.25), we have

$G(g u, g u, f u) \leq \alpha G(g u, g u, fu)$,

a contradiction by considering $\alpha \in (0, 1)$.

Hence it follows that $G(g u, g u, fu) = 0$.

Therefore $g u = f u = w$. (2.1.28)

Hence f and g have a coincidence point.

Since f and g are weakly compatible, we have f f u = g g u

$$\text{i.e., } f\omega = g\omega. \tag{2.1.29}$$

By condition (vi) of our assumption, it follows that $g\,u \preccurlyeq g\,(g\,u) = g\,\omega$.

Now, on using the inequality (2.1.1) with x= y = u and z =ω, and from (2.1.29), we have

$$\begin{aligned} G(f\,u, f\,u, f\,\omega) &\le \alpha \max\{ G(f\,u, g\,u, g\,\omega), G(g\,u, f\,u, g\,w), G(g\,u, g\,u, g\omega), \\ &\qquad G(g\,u, g\,u, f\,\omega), G(g\,u, f\,u, f\,u), G(f\,u, f\,u, g\,u)\} \\ &\quad + L \min\{ G(f\,u, f\,u, g\,\omega), \quad G(g\,u, f\,u, g\,w), \ G(g\,u, g\,u, fu)\} \\ &\le \alpha \max\{ G(f\,u, f\,u, f\,\omega), 0, G(f\,u, f\,u, f\,\omega), G(f\,u, f\,u, f\,\omega), 0, 0 \} \\ &\quad + L \min\{ G(f\,u, f\,u, g\,\omega), \quad G(g\,u, f\,u, g\,w), \ 0\} \\ &= G(f\,u, f\,u, f\,\omega). \end{aligned}$$

Thus $\omega = f\,u = fw$.

Hence, from (2.1.29), it follows that $f\,\omega = g\,\omega = \omega$, i.e., ω is a common fixed point of f and g.

The following example shows that that if any two elements of the fixed points of f and g are not comparable with any element of X then the uniqueness of the fixed point fails.

Example 2.2: Let X = {(0, 1), (1, 2), (-1, 0)}. Define $\precsim$ on X by $(a, b) \precsim (c,d)$ if and only if $a \le c$ and $b \ge d$. Clearly, $(X,\precsim)$ is a poset. Let G be the G- metric on X × X × X defined as

$G(x, y, z) = |x-y| + |y-z| + |z-x|$, for all x, y, z ∈ X. Then (X, G) is a G- metric space.

We define f, g: X × X → X × X by

$$f(x, y) = \begin{cases} (0,1) & if\ (x,y) = (0,1) \\ (1,2) & if\ (x,y) \in \{(1,2),(-1,0)\} \end{cases}$$

and

$$g(x, y) = \begin{cases} (0,1) & if\ (x,y) \in \{(0,1),(-1,0)\} \\ (1,2) & if\ (x,y) \in \{(1,2)\}. \end{cases}$$

The only comparable elements in X are (x, x) for all x ∈ X.

Hence condition (2.1.1) holds trivially since G (f x, f x, f x) = 0. Also f X ⊆ g X, g X is closed and f is g- non decreasing. Further, g (0, 1) =f (0, 1) and g (1, 2) =f (1, 2) . All the conditions of Theorem 2.1 are satisfied, f and g have two common fixed points (0, 1) and (1, 2).

Hence, we observe that if any two elements of the fixed points of f and g are not comparable with any element of X then the uniqueness of the fixed point fails.

If L=0, we have the following corollary.

Corollary 2.3: Let (X, $\preccurlyeq$) be a partially ordered set and suppose that there is a G metric on X such that (X , G) is a complete G- metric space . Let f, g: X → X be two self maps on X satisfying the following the conditions:

(i) f (X) ⊆ g (X);

(ii) g (X) is closed;

(iii) f is g – non decreasing mapping;

(iv) There exists a constant $\alpha \in (0, 1)$ such that for all x ,y z ∈ X with $gx \succcurlyeq gy \succcurlyeq gz$,

$$G(f\,x, f\,y, f\,z) \le \alpha \max\{G(f\,x, g\,y, g\,z), G(g\,x, f\,y, g\,z), G(g\,x, g\,y, g\,z), G(g\,x, g\,y, f\,z),$$

$$G(g\,x, f\,x, f\,x), G(f\,x, f\,y, g\,x)\} \quad (2.3.1)$$

(v) there exists an $x_0 \in X$ with $g\,x_0 \preccurlyeq f\,x_0$;

(vi) $\{g(x_n)\} \subseteq X$ is a non decreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then

$g\,x_n \preccurlyeq g\,z$, $g\,z \preccurlyeq g(g\,z)$ for all n holds.

Then f and g have a coincidence point, *i.e.*, there exists $z \in X$ such that fz=gz. Further, if f and g are weakly compatible, then f and g have a common fixed point.

Theorem 2.4: In addition to the hypothesis of Theorem 2.1 suppose that for all x, u in X there is a in X such that f a is comparable to f x and f u. Then f and g have a unique common fixed point x in X.

Proof: In view of Theorem 2.1, the set of fixed points of f and g is non empty. Suppose that x and u are two common fixed points of f and g.

$$i.e.,\ f\,x = g\,x = x \quad (2.4.1)$$

$$f\,u = g\,u = u. \quad (2.4.2)$$

We now show that f x = f u.

By our assumption there exists a in X such that f a is comparable to f x and f u.

Without loss of generality assume that

$$f\,x \precsim f\,a \text{ and } g\,x \precsim g\,a \quad (2.4.3)$$

Let $a_0 = a$. Since $f\,X \subseteq g\,X$, there exists a_1 in X such that $f\,a_0 = ga_1$, on continuing this process we obtain sequence $\{g\,a_n\}$ such that $g\,a_n = f\,a_{n-1}$ for all n. (2.4.4)

By applying the proof of Theorem 2.1, it follows that $\{g\,a_n\}$ is a G-Cauchy sequence in X, since g X is complete, there exists z in X such that

$$\lim_{n\to\infty} ga_n = g\,z. \quad (2.4.5)$$

We now show that $g\,x \precsim g\,a_n$ for all n.

By (2.4.3), we have $f\,x \precsim f\,a = f\,a_0 = g\,a_1$. (2.4.6)

Hence from (2.4.1) and (2.4.6), it follows that $g\,x \precsim g\,a_1$.

Since f is g- non decreasing, we obtain $f\,x \precsim f\,a_1$. In view of (2.4.4), we have $f\,x \precsim f\,a_1 = g\,a_2$

This implies $g\,x \precsim g\,a_2$.

Recursively, we get $g\,x \precsim g\,a_n$ for all n. (2.4.7)

Since from (2.4.7), we have $g\,x \precsim g\,a_n$, using the inequality (2.1.1), we have

$$\begin{aligned} G(ga_{n+1}, ga_{n+1}, g\,x) &= G(f\,a_n, f\,a_n, f\,x) \\ &\le \alpha \max\{G(f\,a_n, g\,a_n, g\,x), G(g\,a_n, f\,a_n, g\,x), G(g\,a_n, g\,a_n, g\,x), G(g\,a_n, g\,a_n, f\,x), \\ &\qquad G(g\,a_n, f\,a_n, f\,a_n), G(f\,a_n, f\,a_n, g\,a_n)\} \\ &\quad + L\min\{G(f\,a_n, f\,a_n, g\,x), G(g\,a_n, f\,a_n, g\,x), G(ga_n, g\,a_n, fa_n)\}. \end{aligned}$$

Letting $n \to \infty$ using (2.4.4) and (2.4.5), we have

$$\begin{aligned} G(g\,z, g\,z, f\,x) \le \alpha \max\{&G(g\,z, g\,z, g\,x), G(g\,z, g\,z, g\,x), G(g\,z, g\,z, g\,x), G(g\,z, g\,z, f\,x), \\ &G(g\,z, g\,z, g\,z), G(g\,z, g\,z, g\,z)\} \\ &+ L\min\{G(g\,z, g\,z, g\,z), G(g\,z, g\,z, g\,z), G(g\,z, g\,z, g\,z)\}. \end{aligned}$$

This implies $G(g\,z, g\,z, f\,x) \le \alpha\, G(g\,z, g\,z, g\,x) = \alpha\, G(g\,z, g\,z, f\,x)$

Hence it follows that $g\,z = f\,x = gx$. (2.4.8)

Similarly, we can show that $g\,u \precsim g\,a_n$ for all n.

Again, on using (2.1.1), with $x = a_n$, $y = a_n$ and $z = u$, we get $g\,z = g\,u$. (2.4.9)
Hence from (2.4.8) and (2.4.9), we can conclude that $g\,x = g\,u$.

Thus f and g have a unique common fixed point in X.

Remark 2.5: In addition to the conditions of Corollary 2.3, for all x, y in X, there exists a in X such that fa is comparable to fx and f y , then f and g have a unique fixed point in X.

Corollary 2.6: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a G metric on X such that

(X , G) is a complete G- metric space . Let f, g: $X \to X$ be two self maps on X satisfying the following the conditions:

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ is closed;

(iii) f is g – non decreasing mapping;

(iv) there exists a constant $\alpha \in (0, 1)$ such that for all $x, y, z \in X$ with $gx \succcurlyeq gy$,

$$G(fx, fy, fy) \le \alpha \max\{G(fx, gy, gy), G(gx, fy, gy), G(gx, gy, gy), G(gx, fx, fx), G(fx, fy, gx)\} + L \min\{G(fx, fx, gy), G(gx, fx, gy), G(gx, gy, fx)\}; \quad (2.6.1)$$

(v) There exists an $x_0 \in X$ with $g\,x_0 \preccurlyeq f\,x_0$;

(vi) $\{g(x_n)\} \subseteq X$ is a non decreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then

$g\,x_n \preccurlyeq g\,z$, $g\,z \preccurlyeq g(g\,z)$ for all n holds.

Then f and g have a coincidence point, *i.e.*, there exists $z \in X$ such that fz=gz. Further, if f and g are weakly compatible, then f and g have a common fixed point. Further, if there an element a in X such that fa is comparable to fu and fx, then f and g have a unique common fixed point in X , If gx=Ix, we have the following two corollaries.

Corollary 2.7: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a G metric on X such that (X , G) is a complete G- metric space . Let f: $X \to X$ be a mapping on X satisfying the following the conditions:

(i) there exists a constant $\alpha \in (0, 1)$ such that for all $x, y, z \in X$ with $x \succcurlyeq y \succcurlyeq z$,

$$G(fx, fy, fz) \le \alpha \max\{G(fx, y, z), G(x, fy, z), G(x, y, z), G(x, y, fz), G(x, fx, fx), G(fx, fy, x)\} + L \min\{G(fx, fx, z), G(x, fx, z), G(x, y, fx)\} \quad (2.7.1)$$

(ii) There exists an $x_0 \in X$ with $x_0 \preccurlyeq f\,x_0$.

(iii) $\{x_n\} \subseteq X$ is a non decreasing sequence with $x_n \to z$, then $x_n \preccurlyeq z$, for all n holds.

Then f has a fixed point in X. Further, if there an element a in X which is comparable to fu and fx, then f has a unique fixed point in X.

Corollary 2.8: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a G metric on X such that (X , G) is a complete G- metric space . Let f: $X \to X$ be a mapping on X satisfying the following the conditions:

(i) there exists a constant $\alpha \in (0, 1)$ such that for all $x, y, z \in X$ with $x \succcurlyeq y \succcurlyeq z$,

$$G(fx, fy, fz) \le \alpha \max\{G(fx, y, z), G(x, fy, z), G(x, y, z), G(x, y, fz), G(x, fx, fx), G(fx, fy, x)\} \quad (2.8.1)$$

(ii) There exists an $x_0 \in X$ with $x_0 \preccurlyeq f\,x_0$

(iii) $\{x_n\} \subseteq X$ is a non decreasing sequence with $x_n \to z$, then $x_n \preccurlyeq z$, for all n holds.

Then f has a fixed point in X. Further, if there an element a in X such that fa is comparable to fu and fx, then f has a unique fixed point in X.

The following example supports our theorem.

Example 2.9 : Let X = [0, 1] with the usual ordering and we define G on X given by

G (x, y, z) = $|x-y|+|y-z|+|z-x|$, for all x, y, z ∈ X. We define f, g: X → X by

$$f(x) = \frac{x^3}{3}, \; g(x) = \frac{3x^3}{8} \quad \text{for all x in X.}$$

Clearly f X = $[0, \frac{1}{3}] \subseteq [0, \frac{3}{8}]$ f is g- non decreasing and g(X) is closed.

Let x, y, z ∈ X be arbitrary. We say that x > y > z implies x ≥ y ≥ z, we have $\frac{3}{8}x^3 \geq \frac{3}{8}y^3 \geq \frac{3}{8}z^3$, which yields g x ≥ g y ≥ g z.

Now we verify the inequality (2.1.1) with $\alpha = \frac{8}{9}$ and L=0. Since g x ≥ g y ≥ g z, we have

$$\begin{aligned} G(f x, f y, f z) &= \frac{1}{3}[|x^3-y^3|+|y^3-z^3|+|z^3-x^3|] \\ &= \frac{8}{9} \cdot \frac{3}{8}[|x^3-y^3|+|y^3-z^3|+|z^3-x^3|] \\ &\leq \frac{8}{9} G(gx, gy, gz) \\ &\leq \frac{8}{9} \max\{G(f x, g y, g y), G(g x, f y, g y), G(g x, g y, g y), G(g x, g y, f y), \\ &\qquad G(g x, f x, f x), G(f x, f Y, g x)\}. \end{aligned}$$

Hence the inequality (2.1.1) holds for all x, y, z in X and '0' is the unique common fixed point of f and g.

Example 2.10: Let X = [0, 2]. Then (X, ≤) is a poset. We define G: X × X × X → $\mathbb{R}^+$ by

G (x, y, z) = max { $|x-y|$, $|y-z|$, $|z-x|$ } for all x, y, z in X. We define f, g: X → X by

$$f x = \begin{cases} 0 & if\ x \in [0,1] \cup \{2\} \\ 2 & if\ x \in (1,2) \end{cases} \quad \text{and} \quad g x = \begin{cases} 0\ if\ x \in \{0,2\} \\ 1\ if\ x \in (0,1] \\ 2\ if\ x \in (1,2). \end{cases}$$

Clearly f X = { 0,2} ⊆ { 0, 1, 2} = g X, g X is closed and f is g – non decreasing .

We consider x, y, z ∈ X such that g x ≤ g y ≤ g z. The inequality (2.1.1) satisfies with $\alpha = \frac{1}{2}$ and L = 2 .

Here, we observe that condition (2.3.1) fails to holds for any $\alpha \in (0,1)$.

Hence Theorem 2.1 generalizes Corollary 2.3.

Also, the inequality (1.13.1) fails to holds by choosing x=1, y= 1 and z=$\frac{3}{2}$, since

G(fx,fy,fz)=G(0, 0, 2) = 2 $\nleq \alpha$ (1)= α G(gx,gy,gz),

for any $\alpha \in (0,1)$.

Thus, Theorem 2.4 is a generalization of Theorem 1.13.

The following example shows that Theorem 2.4 is a generalization of Corollary 2.6 and if we transform a metric in to G-metric, it is not equivalent to condition (1.2) of [1] in a metric space.

Example 2.11: Let X = [0, 1] and x ≼ y implies x ≥ y and we define d on X by d(x, y) = $|x-y|$. We define G on X by G(x, y, z) = max { $|x-y|, |y-z|, |z-x|$ } . Clearly (X , ≼ , G) is a complete partially ordered G- metric space and (X , ≤ , d) is a complete partially ordered metric space . We now define f and g on X by

$$f(x) = \begin{cases} x + \frac{1}{8} & if\ 0 \le x \le \frac{1}{4} \\ \frac{5x}{6} + \frac{1}{6} & if\ \frac{1}{4} \le x \le 1 \end{cases} \qquad g(x) = x \text{ for all } x \in X.$$

Clearly, f is g non – decreasing and f X ⊆ g X , g X is closed . We will verify the inequality (2.1.1) with $\alpha = \frac{1}{16}$ and L = 1.

Let g x ≼ g y ≼ g z. Then

G (f x, f y, f z) = max { $|fx - fy|$, $|fy - fz|$, $|fx - fz|$ }, for all gx ≥ gy ≥ gz .

As f is g- non decreasing, we have G (f x, f y, f z) = $|fx - fz|$.

Also,

max { G(f x, g y, g z), G(g x, f y, g z), G(g x, g y, g z), G(g x, g y ,f z), G(g x, f x, f x),

G(f x, f y, g x) }

= max { $|fx - y|, |y - z|, |fx - z|, |x - fy|, |fy - z|, |x - z|, |x - y|, |y - fz|, |fz - x|,$

$|x - fx|, |fx - fy|2$ } .

Again f is g- non decreasing, we have

$|fx - z| \le$ max { $|fx - y|, |y - z|, |fx - z|, |x - fy|, |fy - z|, |x - z|, |x - y|, |y - fz|$

$|x - fx|, |fx - fy|$ } $\le |1 - z|$

And L min { $|fx - fx|, |fx - z|, |x - fx|, |x - z|, |x - y|, |y - fx|$ } = L (0) = 0 .

Hence, we have

α max { $|fx - y|, |y - z|, |fx - z|, |x - fy||fy - z|, |x - z|, |x - y|, |y - fz|, |fz - x|, |x - fx|, |fx - fy|$ } + L min { $|fx - fz|, |fx - z|, |x - fx|, |x - z|, |x - y|, |y - fx|$} -G(fx,fy,fz) $\ge |fx - z| - |fx - fz| = |fz - z| \ge 0$ since fz ≥ z.

Let

α max { $|fx - y|, |y - z|, |fx - z|, |x - fy||fy - z|, |x - z|, |x - y|, |y - fz|, |fz - x|, |x - fx|, |fx - fy|$ } + L min { $|fx - fz|, |fx - z|, |x - fx|, |x - z|, |x - y|, |y - fx|$} -

G (f x, f y, f z) = P.

$$\text{Hence } P \ge \begin{cases} \frac{1}{8}\ if\ 0 \le z \le \frac{1}{4} \\ \frac{1-z}{16}\ \ if\ \frac{1}{4} \le z\, 1\,. \end{cases}$$

Thus p $\ge \frac{1-z}{16} + 1(0)$.

Hence (2.1.1) holds with $\alpha = \frac{1}{16}$ and L = 1. Also 1 is a fixed point of f and g.

Now, the inequality (2.6.1) fails at $x = \frac{9}{64}$, $y = \frac{1}{8}$, $z = \frac{1}{8}$. Indeed, we have

$$G(f x, f y, f y) = G\left(\frac{17}{64}, \frac{1}{16}, \frac{1}{16}\right) = \frac{13}{64}$$

$$\le \alpha \max\left\{\left|\frac{17}{64} - \frac{1}{8}\right|, 0, \left|\frac{17}{64} - \frac{1}{8}\right|, \left|\frac{9}{64} - \frac{1}{16}\right|, \left|\frac{1}{16} - \frac{1}{8}\right|, \left|\frac{9}{64} - \frac{1}{8}\right|, \left|\frac{9}{64} - \frac{1}{8}\right|, \left|\frac{1}{8} - \frac{1}{16}\right|, \left|\frac{1}{16} - \frac{9}{64}\right|, \left|\frac{9}{64} - \frac{17}{64}\right|, \left|\frac{17}{64} - \frac{1}{16}\right|\right\}$$

$$+ L \min\{0\}$$

$$= \alpha \max\left\{\frac{9}{64}, 0, \frac{9}{64}, \frac{5}{64}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{16}, \frac{5}{64}, \frac{8}{64}, \frac{9}{64}\right\}$$

$G(f x, f y, f y) = \frac{13}{64} \nleq \alpha\, \frac{9}{64} = \alpha$ max {G (f x, g y, f y), G (g x, f y, g y), G(g x, g y, g y),

$$G(g\,x, g\,y, f\,y),\ \ G(g\,x, f\,x, f\,x), G(f\,x, f\,y, g\,x) \}$$

$$+ L \min\{G\,(f\,x, f\,y, g\,y), G(g\,x, f\,x, g\,y), G(g\,x, g\,y, f\,x)\}$$

for any $\alpha \geq 0$ with $\alpha < 1$ and L >0.

Hence Theorem 2.4 is a generalization of Corollary 2.6.

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